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AN EPIMORPHISM BETWEEN KNOT GROUPS WHICH DOES NOT MAP A MERIDIAN TO A MERIDIAN

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1. Introduction

Let K be a knot in S^3 and $G(K)$ the knot group. The existence of an epimorphism between knot groups defines a partial order on the set of prime knots. This partial order on the set of prime knots with up to 10 crossings is determined in [4]. The result of [4] is extended to prime knots with up to 11 crossings in [2]. The key criterion to determine that there exists no epimorphism between given knot groups is an application of the main result of [5]. On the other hand, an epimorphism for each pair of knots which admits an epimorphism is given explicitly in [4] and [2], in order to show the existence of an epimorphism. In their papers [4] and [2], all the epimorphisms map meridians to meridians. In this paper, we show an example of an epimorphism which does not map a meridian to a meridian.

2. Definition of an epimorphism and main theorem

Let K_1, K_2 be knots as depicted in Figure 1 and Figure 2 respectively.

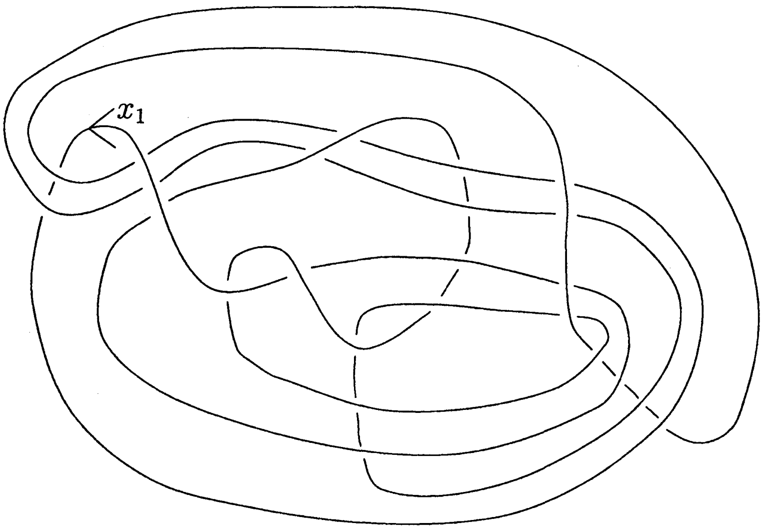

FIGURE 1. Knot K_1

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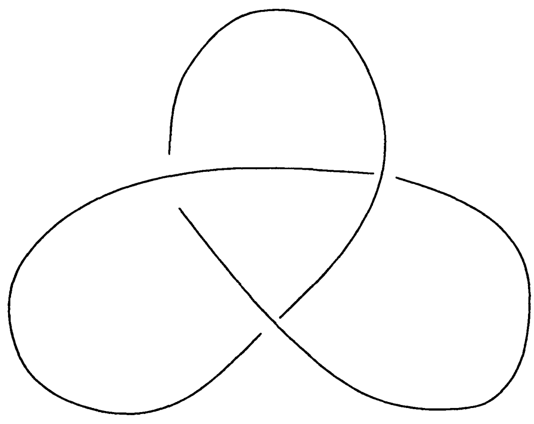

FIGURE 2. Knot K_2

The knot group $G(K_1)$ admits a Wirtinger presentation with generators $x_1, x_2, \dots, x_{24}$ and defining relators:

$x_6x_2\bar{x}_6\bar{x}_1$,	$x_{10}x_2\bar{x}_{10}\bar{x}_3$,	$x_6x_3\bar{x}_6\bar{x}_4$,	$x_{22}x_4\bar{x}_{22}\bar{x}_5$,	$x_1x_6\bar{x}_1\bar{x}_5$,	$x_{17}x_7\bar{x}_{17}\bar{x}_6$,
$x_{23}x_7\bar{x}_{23}\bar{x}_8$,	$x_{13}x_9\bar{x}_{13}\bar{x}_8$,	$x_3x_9\bar{x}_3\bar{x}_{10}$,	$x_1x_{10}\bar{x}_1\bar{x}_{11}$,	$x_{22}x_{12}\bar{x}_{22}\bar{x}_{11}$,	$x_6x_{13}\bar{x}_6\bar{x}_{12}$,
$x_{23}x_{14}\bar{x}_{23}\bar{x}_{13}$,	$x_{17}x_{14}\bar{x}_{17}\bar{x}_{15}$,	$x_{18}x_{16}\bar{x}_{18}\bar{x}_{15}$,	$x_6x_{17}\bar{x}_6\bar{x}_{16}$,	$x_1x_{17}\bar{x}_1\bar{x}_{18}$,	$x_{16}x_{19}\bar{x}_{16}\bar{x}_{18}$,
$x_{24}x_{19}\bar{x}_{24}\bar{x}_{20}$,	$x_{12}x_{21}\bar{x}_{12}\bar{x}_{20}$,	$x_4x_{21}\bar{x}_4\bar{x}_{22}$,	$x_1x_{23}\bar{x}_1\bar{x}_{22}$,	$x_6x_{23}\bar{x}_6\bar{x}_{24}$,	$x_{18}x_{24}\bar{x}_{18}\bar{x}_1$,

where $\bar{x}_i = {x_i}^{-1}$. All the generators are conjugate to one another and we can regard x_1 as a meridian of K_1.

The knot K_2 is called the trefoil and the knot group $G(K_2)$ admits a presentation:

$$G(K_2) = \langle y_1, y_2 \mid y_1y_2y_1 = y_2y_1y_2 \rangle.$$

We define a map $f : G(K_1) \to G(K_2)$ by the following. Here we write numbers $1, 2$ for the generators y_1, y_2 respectively. For example, $12\bar{1}2\bar{1}$ means $y_1y_2{y_1}^{-1}y_2{y_1}^{-1}$.

$$
\begin{array}{ll}
f(x_1) = 12\bar{1}2\bar{1}, & f(x_2) = 1\bar{2}1\bar{2}1\bar{2}\bar{1}222\bar{1}2\bar{1}2\bar{1}, \\
f(x_3) = 12\bar{1}2\bar{1}, & f(x_4) = 1\bar{2}12\bar{1}2\bar{1}21\bar{2}\bar{1}2\bar{1}, \\
f(x_5) = 212\bar{1}2\bar{1}21\bar{2}\bar{1}\bar{2}, & f(x_6) = 1\bar{2}12\bar{1}2\bar{1}2\bar{1}, \\
f(x_7) = 1\bar{2}1\bar{2}\bar{2}122\bar{1}\bar{1}222\bar{1}2\bar{1}, & f(x_8) = 1\bar{2}12\bar{1}2\bar{1}2\bar{1}, \\
f(x_9) = 1\bar{2}1\bar{2}\bar{2}12\bar{1}2\bar{1}22\bar{1}2\bar{1}, & f(x_{10}) = 1\bar{2}12\bar{1}2\bar{1}2\bar{1}, \\
f(x_{11}) = 122\bar{1}\bar{1}, & f(x_{12}) = 1\bar{2}\bar{2}122\bar{1}\bar{1}22\bar{1}, \\
f(x_{13}) = 12\bar{1}2\bar{1}, & f(x_{14}) = 1\bar{2}1\bar{2}\bar{2}\bar{2}12\bar{1}2\bar{1}222\bar{1}2\bar{1}, \\
f(x_{15}) = 12\bar{1}2\bar{1}, & f(x_{16}) = 1\bar{2}\bar{2}12\bar{1}2\bar{1}22\bar{1}, \\
f(x_{17}) = 1\bar{2}1\bar{2}1\bar{2}\bar{1}\bar{2}12\bar{1}2\bar{1}212\bar{1}2\bar{1}2\bar{1}, & f(x_{18}) = 22\bar{1}, \\
f(x_{19}) = 1\bar{2}\bar{2}1\bar{2}1\bar{2}\bar{1}222\bar{1}2\bar{1}22\bar{1}, & f(x_{20}) = 22\bar{1}, \\
f(x_{21}) = 1\bar{2}12\bar{1}\bar{2}1\bar{2}1\bar{2}\bar{1}2\bar{1}212\bar{1}2\bar{1}21\bar{2}\bar{1}2\bar{1}, & f(x_{22}) = 22\bar{1}, \\
f(x_{23}) = 1\bar{2}1\bar{2}\bar{1}222\bar{1}2\bar{1}, & f(x_{24}) = 1\bar{2}12\bar{1}\bar{1}221\bar{2}\bar{1}2\bar{1}.
\end{array}
$$

Theorem 2.1. *The above mapping $f : G(K_1) \to G(K_2)$ is an epimorphism which does not map a meridian of K_1 to a meridian of K_2.*

3. Proof

In this section, we show Theorem 2.1. First, we will check the relators of $G(K_1)$ vanish under the mapping f.

$$
\begin{aligned}
f(x_6x_2\bar{x}_6\bar{x}_1) &= 1\bar{2}12\bar{1}2\bar{1}2\bar{1}1\bar{2}1\bar{2}1\bar{2}\bar{1}222\bar{1}2\bar{1}2\bar{1}1\bar{2}1\bar{2}1\bar{2}\bar{1}2\bar{1}1\bar{2}1\bar{2}\bar{1} = e,\\
f(x_{10}x_2\bar{x}_{10}\bar{x}_3) &= 1\bar{2}12\bar{1}2\bar{1}2\bar{1}1\bar{2}1\bar{2}1\bar{2}\bar{1}222\bar{1}2\bar{1}2\bar{1}1\bar{2}1\bar{2}1\bar{2}\bar{1}2\bar{1}1\bar{2}1\bar{2}\bar{1} = e,\\
f(x_6x_3\bar{x}_6\bar{x}_4) &= 1\bar{2}12\bar{1}2\bar{1}2\bar{1}12\bar{1}2\bar{1}1\bar{2}1\bar{2}1\bar{2}\bar{1}2\bar{1}1\bar{2}12\bar{1}\bar{2}1\bar{2}1\bar{2}\bar{1}2\bar{1} = e,\\
f(x_{22}x_4\bar{x}_{22}\bar{x}_5) &= 22\bar{1}1\bar{2}12\bar{1}2\bar{1}21\bar{2}\bar{1}2\bar{1}1\bar{2}\bar{2}212\bar{1}\bar{2}1\bar{2}1\bar{2}\bar{1}\bar{2} = e,\\
f(x_1x_6\bar{x}_1\bar{x}_5) &= 12\bar{1}2\bar{1}1\bar{2}12\bar{1}2\bar{1}2\bar{1}1\bar{2}1\bar{2}\bar{1}212\bar{1}\bar{2}1\bar{2}1\bar{2}\bar{1}\bar{2}\\
&= 122\bar{1}\bar{1}212\bar{1}\bar{2}1\bar{2}1\bar{2}\bar{1}\bar{2} = 122\bar{1}\bar{1}121\bar{1}\bar{2}1\bar{2}1\bar{1}\bar{2}\bar{1} = e.
\end{aligned}
$$

The above and similar calculations imply that $f : G(K_1) \to G(K_2)$ is a group homomorphism. Next, we will show that the group homomorphism f is surjective. Since 1 and 2 are generators for $G(K_2)$, it is sufficient to find elements of $G(K_1)$ which are mapped to 1 and 2 under the group homomorphism f.

$$
\begin{aligned}
&f(x_{18}x_6\bar{x}_1\bar{x}_1x_{18}x_6\bar{x}_1)\\
&\quad= f(x_{18})f(x_6)\overline{f(x_1)}\,\overline{f(x_1)}f(x_{18})f(x_6)\overline{f(x_1)}\\
&\quad= 22\bar{1}1\bar{2}12\bar{1}2\bar{1}2\bar{1}1\bar{2}1\bar{2}\bar{1}1\bar{2}1\bar{2}\bar{1}22\bar{1}1\bar{2}12\bar{1}2\bar{1}2\bar{1}1\bar{2}1\bar{2}\bar{1}\\
&\quad= 212\bar{1}\bar{2}1\bar{2}\bar{1}212\bar{1}\bar{1} = 121\bar{1}\bar{2}1\bar{2}\bar{1}121\bar{1}\bar{1} = 1,\\
&f(x_1\bar{x}_6\bar{x}_{18}x_1x_{18}x_6\bar{x}_1\bar{x}_1x_{18}x_6\bar{x}_1)\\
&\quad= f(x_1)\overline{f(x_6)}\,\overline{f(x_{18})}f(x_1)f(x_{18})f(x_6)\overline{f(x_1)}\,\overline{f(x_1)}f(x_{18})f(x_6)\overline{f(x_1)}\\
&\quad= 12\bar{1}2\bar{1}1\bar{2}1\bar{2}\bar{1}2\bar{1}1\bar{2}\bar{2}12\bar{1}2\bar{1}22\bar{1}1\bar{2}12\bar{1}2\bar{1}2\bar{1}1\bar{2}1\bar{2}\bar{1}1\bar{2}1\bar{2}\bar{1}22\bar{1}1\bar{2}12\bar{1}2\bar{1}2\bar{1}1\bar{2}1\bar{2}\bar{1}\\
&\quad= 11\bar{2}\bar{1}\bar{2}12\bar{1}2\bar{1}212\bar{1}\bar{2}1\bar{2}\bar{1}212\bar{1}\bar{1} = 11\bar{1}\bar{2}112\bar{1}\bar{2}1121\bar{1}\bar{2}1\bar{2}1121\bar{1}\bar{1} = 2.
\end{aligned}
$$

Therefore it is shown that the group homomorphism f is surjective. Finally, we will prove that f does not map a meridian of K_1 to a meridian of K_2. We can fix meridians for K_1 and K_2 by x_1 and 1, without loss of generality. An $SL(2;\mathbb{Z})$-representation ρ of $G(K_2)$ is defined by

$$\rho(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \rho(2) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

We can check easily that ρ is a representation of $G(K_2)$. Note that the trace of $\rho(1)$ is 2. On the other hand, we get

$$\rho(f(x_1)) = \rho(12\bar{1}2\bar{1}) = \begin{pmatrix} -1 & 2 \\ -3 & 5 \end{pmatrix}.$$

Then the trace of $\rho(f(x_1))$ is not equal to 2. Hence $f(x_1)$ is not conjugate to 1. It follows that the epimorphism f does not map a meridian of K_1 to a meridian of K_2. This completes the proof.

4. Problem

In this section, we propose a problem related to epimorphisms between knot groups. We review the result of the partial order with respect to the existence of an epimorphism on the set of prime knots with up to 11 crossings.

Theorem 4.1 (Kitano-Suzuki [4], Horie-Kitano-Matsumoto-Suzuki [2]). *The following pairs admit epimorphisms between their knot groups, which map meridians to meridians:*

$(8_5, 3_1), (8_{10}, 3_1), (8_{15}, 3_1), (8_{18}, 3_1), (8_{19}, 3_1), (8_{20}, 3_1), (8_{21}, 3_1),$
$(9_1, 3_1), (9_6, 3_1), (9_{16}, 3_1), (9_{23}, 3_1), (9_{24}, 3_1), (9_{28}, 3_1), (9_{40}, 3_1),$
$(10_5, 3_1), (10_9, 3_1), (10_{32}, 3_1), (10_{40}, 3_1), (10_{61}, 3_1), (10_{62}, 3_1), (10_{63}, 3_1), (10_{64}, 3_1),$
$(10_{65}, 3_1), (10_{66}, 3_1), (10_{76}, 3_1), (10_{77}, 3_1), (10_{78}, 3_1), (10_{82}, 3_1), (10_{84}, 3_1), (10_{85}, 3_1),$
$(10_{87}, 3_1), (10_{98}, 3_1), (10_{99}, 3_1), (10_{103}, 3_1), (10_{106}, 3_1), (10_{112}, 3_1), (10_{114}, 3_1), (10_{139}, 3_1),$
$(10_{140}, 3_1), (10_{141}, 3_1), (10_{142}, 3_1), (10_{143}, 3_1), (10_{144}, 3_1), (10_{159}, 3_1), (10_{164}, 3_1),$
$(11a_{43}, 3_1), (11a_{44}, 3_1), (11a_{46}, 3_1), (11a_{47}, 3_1), (11a_{57}, 3_1), (11a_{58}, 3_1), (11a_{71}, 3_1),$
$(11a_{72}, 3_1), (11a_{73}, 3_1), (11a_{100}, 3_1), (11a_{106}, 3_1), (11a_{107}, 3_1), (11a_{108}, 3_1), (11a_{109}, 3_1),$
$(11a_{117}, 3_1), (11a_{134}, 3_1), (11a_{139}, 3_1), (11a_{157}, 3_1), (11a_{165}, 3_1), (11a_{171}, 3_1), (11a_{175}, 3_1),$
$(11a_{176}, 3_1), (11a_{194}, 3_1), (11a_{196}, 3_1), (11a_{203}, 3_1), (11a_{212}, 3_1), (11a_{216}, 3_1), (11a_{223}, 3_1),$
$(11a_{231}, 3_1), (11a_{232}, 3_1), (11a_{236}, 3_1), (11a_{244}, 3_1), (11a_{245}, 3_1), (11a_{261}, 3_1), (11a_{263}, 3_1),$
$(11a_{264}, 3_1), (11a_{286}, 3_1), (11a_{305}, 3_1), (11a_{306}, 3_1), (11a_{318}, 3_1), (11a_{332}, 3_1), (11a_{338}, 3_1),$
$(11a_{340}, 3_1), (11a_{351}, 3_1), (11a_{352}, 3_1), (11a_{355}, 3_1), (11n_{71}, 3_1), (11n_{72}, 3_1), (11n_{73}, 3_1),$
$(11n_{74}, 3_1), (11n_{75}, 3_1), (11n_{76}, 3_1), (11n_{77}, 3_1), (11n_{78}, 3_1), (11n_{81}, 3_1), (11n_{85}, 3_1),$
$(11n_{86}, 3_1), (11n_{87}, 3_1), (11n_{94}, 3_1), (11n_{104}, 3_1), (11n_{105}, 3_1), (11n_{106}, 3_1), (11n_{107}, 3_1),$
$(11n_{136}, 3_1), (11n_{164}, 3_1), (11n_{183}, 3_1), (11n_{184}, 3_1), (11n_{185}, 3_1),$
$(8_{18}, 4_1), (9_{37}, 4_1), (9_{40}, 4_1),$
$(10_{58}, 4_1), (10_{59}, 4_1), (10_{60}, 4_1), (10_{122}, 4_1), (10_{136}, 4_1), (10_{137}, 4_1), (10_{138}, 4_1),$
$(11a_5, 4_1), (11a_6, 4_1), (11a_{51}, 4_1), (11a_{132}, 4_1), (11a_{239}, 4_1), (11a_{297}, 4_1), (11a_{348}, 4_1),$
$(11a_{349}, 4_1), (11n_{100}, 4_1), (11n_{148}, 4_1), (11n_{157}, 4_1), (11n_{165} 4_1),$
$(11n_{78}, 5_1), (11n_{148}, 5_1),$
$(10_{74}, 5_2), (10_{120}, 5_2), (10_{122}, 5_2), (11n_{71}, 5_2), (11n_{185}, 5_2),$
$(11a_{352}, 6_1),$
$(11a_{351}, 6_2),$
$(11a_{47}, 6_3), (11a_{239}, 6_3).$

The other pairs of prime knots with up to 11 *crossings do not admit any epimorphism sending a meridian to a meridian.*

In this table, the numbering of the knots with up to 11 *crossings follows that of the web page "KnotInfo"* [1], *which is operated by Cha and Livingston.*

We can see all the epimorphisms for the pairs of Theorem 4.1, in [4] and [2]. As mentioned in Section 1, each of them maps a meridian to a meridian.

Problem 4.2. *Which pair of Theorem 4.1 admit an epimorphism between their knot groups which does not map a meridian to a meridian? In particular, does there exist such an epimorphism between* 2*-bridge knot groups?*

We note that Ohtsuki-Riley-Sakuma [7] and Lee-Sakuma [6] studied epimorphisms between 2-bridge link groups.

Remark 4.3. The Alexander polynomial of K_1 is $t^4 - 2t^3 + 3t^2 - 2t + 1$. All the prime knots with up to 11 crossings which have the same Alexander polynomials are 8_{20}, 10_{140}, $11n_{73}$ and $11n_{74}$. Moreover, compared with the numbers of $SL(2; \mathbb{Z}/p\mathbb{Z})$-representations of $G(8_{20})$, $G(10_{140})$, $G(11n_{73})$, $G(11n_{74})$ and $G(K_1)$ for $p = 2, 3, 5$, we can conclude that K_1 is not a prime knot with up to 11 crossings. Hence the pair (K_1, K_2) does not appear in Theorem 4.1. In addition, Boileau-Kitano-Morifuji has informed the author that the knot K_1 is a prime knot, since they checked K_1 is a hyperbolic knot by using SnapPea.

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